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On the Lower Bound of Spectrum of the Schrödinger's Operator for Some Multi-Particle Systems

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ABSTRACT

For N - particle Schrödinger's operator with Column potential with central charge equal Z, it is well known estimation $N < 2Z + 1$, obtained by Lieb E. In the present paper the estimation obtained in case when there is a different interaction between particles.

Keywords: Spectrum, Lower Bound, Schrödinger Operator, Multi-Particle System

1. Introduction

In the space $L_2(R^{3N})$ consider

$$H_N = -\Delta + W(x) \tag{1}$$

an operator of interactions of arbitrary $N+1$ particles, where

$$W(x) = \sum_{k=0}^{N} \sum_{j<k} V_{jk}(x_j - x_k), \quad x_j \in R^3 \tag{2}$$

and $V_{jk} = V_{kj}$ when $j = 0, 1, 2, ..., N, \quad k = 0, 1, 2, ..., N$, $\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_N$, Δ_j -three dimensional Laplace operator, $x = (x_1, x_2, ..., x_N) \in R^{3N}$, Ð¸ $x_j = (x_j^1, x_j^2, x_j^3) \in R^3$. Let, $V_k(x_k) = V_{0k}(x_0 - x_k)$, $k = 1, 2, ..., N$. Further put $x_0 = 0$.
Let

$$V(x) = \frac{b(|x|)}{|x|}, x \in R^3 \tag{3}$$

$b(t)$ - is non-negative and non-decreasing function such that $V(x)$ monotonically decreasing with increasing of $|x|$ and $b''(t) < 0$.
Let

$$\alpha_j(x) = -\frac{V_j(x)}{V(x)} \geqslant 0 \tag{4}$$

identically non-zero functions uniformly bounded with respect to $x \in R^3$, where $j = 1, 2, ..., N$. Moreover let

$$\beta_{j\,k}(x) = \frac{V_{j\,k}(x)}{V(x)} \geqslant 0 \tag{5}$$

where $\beta_{jk}(x)$, also identically non-zero and uniformly bounded with respect to $x \in R^3$, where $j = 1, 2, ..., N$, $k = 1, 2, ..., N$

Then above mentioned operator cn be represented as

$$H_N = -\sum_{j=1}^{N} (\Delta_j + \alpha_j(x_j)V(x_j)) + \sum_{k=1}^{N} \sum_{j=1}^{k-1} \beta_{jk}(x_j - x_k)V(x_j - x_k) \tag{6}$$

Let

$$\sigma_1(\alpha) = \frac{1}{N}(\alpha_1(x_1) + \alpha_2(x_2) + \cdots + \alpha_N(x_N)),$$

$$\sigma_2(k,\beta) = \frac{1}{2(N-1)} \sum_{j \neq k} \beta_{jk}(x_j - x_k), \;\; k = 1, 2, ..., N,$$

$$\sigma_3(\beta) = \frac{2}{N(N-1)} \sum_{k=1}^{N} \sum_{j=1}^{k-1} \beta_{jk}(x_j - x_k).$$

Let $E_N = E_N(Z)$ the least eigenvalue of H_N.

2. Main Results.

The following theorems are valid

Theorem 1. *Let for the function $V_{jk}(x), x \in R^3$ in $j \neq k$, $j = 0, 1, 2, \ldots..., N$, $k = 0, 1, 2, \ldots, N$ the following relations hold*

$$\lim_{r\to\infty} \left\{ \sup_{x \in R^3} \frac{1}{r^3} \int\limits_{|x-y| \leqslant r} V_{jk}(|y|) \right\} dy = 0.$$

Then for any $N \geqslant 2$

$$E_N \leqslant E_{N-1}.$$

Theorem 2. *Let there are $Z > 0$ and $\gamma > 0$, both are independent from N, such that*

$$\alpha_k(x_k) \leqslant Z, \;\; \gamma\sigma_2(k,\beta) \geqslant 1, \; k = 1, ..., N$$

uniformly with respect x_i, $x_k \in R^3$, $i = 1, 2, \ldots N$, $k = 1, 2, \ldots N$.
Then there is a number $N_{\max}$ such that for all $N \geqslant N_{\max}$

$$E_N = E_{N_{\max}}.$$

Theorem 3. *Let there are $Z > 0$ and $\gamma > 0$, independent from N, such that*

$$\sigma_1(\alpha) \leqslant Z, \;\; \gamma\sigma_3(\beta) \geqslant 1,$$

uniformly with respect to x_i, $x_k \in R^3$, $i = 1, ..., N$, $k = 1, ..., N$.
Then

$$N_{\max} < 2\gamma Z + 1.$$

3. Remark

In case of Coulomb's potential i.e. when $\alpha_j = Z$, $b \equiv 1$, $\beta_{jk} = 1$, theorems 1 and 2 proved by Ruskai M.B. (Cycon et al., 1987, Ruskai, 1982a,b) and Sigal I.M.(Sigal, 1982, 1984). Theorem 3 in case of Coulomb's potential proved by Lieb E. (Lieb, 1984a,b,c). The case $\alpha_j = 1$, $\beta_{jk} = 1$ all these three theorems proved by Alimov Sh.A. (Alimov, 1992). In (Khalmukhamedov and Kuchkarov, 2003), Khalmukhamedov A.R. and Kuchkarov E.I. obtained these results for the operator

$$H_N = -\sum_{j=1}^{N}\left(\Delta_j + Z\frac{b_1(|x_j|)}{|x_j|}\right) + \sum_{1\leqslant k<j\leqslant N}\frac{b_1(x_j - x_k)}{|x_j - x_k|}$$

and in this case Lieb's estimation has a form

$$N_{\max} < \frac{2}{b_0}Z + 1,$$

where $b_0 = \inf\limits_{t>0}\left(\frac{b_1(t)}{b_2(t)}\right)$.

4. Proof of the Main Results

Before proving Theorem 1 we prove some lemmas.

Lemma 1. *There is a function $\omega_r(x) = \omega_r(|x|) \in C_0^\infty(R^3)$, $r > 0$, such that:*

1) supp $\omega_r(x) \subset \left\{x \in R^3 : |x| \leqslant r+1\right\}$, $\omega_r(x) = C_r\, r^{-\frac{3}{2}}$ *when* $|x| \leqslant r-1$, $c_r = \sqrt{\frac{3}{4\pi}}(1 + O(\frac{1}{\sqrt{r}}))$

when $r \to \infty$;

2) $||\omega_r||_{L_2(R^3)} = 1$;

3) $\lim\limits_{r\to\infty} ||\nabla\omega_r||_{L_2(R^3)} = 0$.

Proof of Lemma 1 Let

$$\omega(t) = \begin{cases} c_0 \exp\left(-\frac{1}{1-t^2}\right), & \text{when } |t| < 1 \\ 0 & \text{when } |t| \geq 1 \end{cases}$$

where $c_0 = \left(\int\limits_{-\infty}^{\infty} \omega(t)dt\right)^{-1}$ is a normalizing constant. Consider a function of the form

$$\omega_R(|x|) = c_R R^{-\frac{3}{2}} \int\limits_{|x|}^{+\infty} \omega(t-R)dt,$$

where, c_R is a normalizing constant, which is determined from the condition $||\omega_R^2(x)||_{L_2(R^3)} = 1$. It is easy to verify that $\lim\limits_{R\to+\infty} c_R = \sqrt{\frac{3}{4\pi}}$.

Let us prove that this function satisfies all the requirements of Lemma 1. In fact, let $|x| \leq R-1$. Then $t-R \geq |x|-R \geq -1$, hence $\omega_R(|x|) = c_R R^{-\frac{3}{2}}$. If $|x| \geq R+1$, then $t-R \geq |x|-R \geq 1$, therefore $\omega_R(|x|) \equiv 0$.

Next

$$|\nabla \omega_R(|x|)|^2 = \sum_{j=1}^{3} \left(\frac{\partial \omega_R(|x|)}{\partial x_j}\right)^2 = c_R^2 R^{-3} \omega^2(|x|-R),$$

then if $R \to +\infty$,

$$||\nabla \omega_R(x)||^2_{L_2(R^3)} = c_R^2 R^{-3} \int\limits_{R-1\leq|x|\leq R+1} \omega^2(|x|-R)dx = O(R^{-1}).$$

Lemma 1 is proved.

Lemma 2. *Let the function $Q(x) \geq 0$, $x \in R^3$, satisfies the relations*

$$\lim_{r\to\infty}\left\{\sup_{x\in R^3} \frac{1}{r^3} \int\limits_{|x-y|\leq r} Q(y)dy\right\} = 0.$$

Then

$$\lim_{r\to\infty}\left\{\sup_{x\in R^3} \frac{1}{r^3} \int\limits_{|x-y|\leq r} Q(y)\omega_r^2(|y|)dy\right\} = 0.$$

Lemma 2 follows directly from Lemma 1.

Proof of Theorem 1.

Let us prove the inequality $E_N \leqslant E_{N-1}$ for every $N \geqslant 2$.

Consider an arbitrary function $\varphi \in C_0^\infty(R^{3N-3})$ such that $||\varphi||_{L_2(R^{3N-3})} = 1$. Let $\psi(x) = \varphi(\tilde{x})\omega_r(x_N)$, where $x = (\tilde{x}, x_N) \in R^{3N-3} \times R^3$. Obviously $\psi \in C_0^\infty(R^{3N})$, as well as$||\psi||_{L_2(R^{3N})} = 1$; Moreover $(H_N\psi, \psi) \geqslant E_N$.

We have

$$(H_N\psi, \psi) = (H_{N-1}\varphi, \varphi)(\omega_r, \omega_r)+$$
$$+(-\Delta_N\omega_r, \omega_r) - (\alpha_N(x_N)V(x_N)\omega_r, \omega_r)+$$

$$+\left(\sum_{k=1}^{N}\sum_{j=1}^{k-1}(\beta_{jk}(x_j - x_k)V(x_j - x_k)\omega_r, \omega_r)\varphi(\tilde{x}), \varphi(\tilde{x})\right)$$

where

$$H_{N-1} = -\sum_{j=1}^{N-1}(\Delta_j + \alpha_j(x_j)V(x_j))+$$
$$+\sum_{k=1}^{N-1}\sum_{j=1}^{k-1}\beta_{jk}(x_j - x_k)V(x_j - x_k)$$

- an operator for $N-1$ particles.

Obviously

$$(H_{N-1}\varphi, \varphi)(\omega_r, \omega_r) = (H_{N-1}\varphi, \varphi) \geqslant E_{N-1}.$$

Applying Lemma 2 obtain

$$(-\Delta_N\omega_r, \omega_r) - (\alpha_N(x_N)V(x_N)\omega_r, \omega_r)+$$

$$+\left(\sum_{k=1}^{N}\sum_{j=1}^{k-1}(\beta_{jk}(x_j - x_k)V(x_j - x_k)\omega_r, \omega_r)\,\varphi(\tilde{x}), \varphi(\tilde{x})\right)$$

$$= o(1)$$

since $(-\Delta_N \omega_r, \omega_r) = ||\nabla \omega_r(x)||^2 = o(1)$,

$$(\alpha_N(x_N)V(x_N)\omega_r, \omega_r) =$$
$$= \int_{R^3} \alpha_N(x_N)V(x_N)\omega_r^2(x_N)dx_N = o(1),$$
$$\int_{R^{3N-1}} |\varphi(\tilde{x})|^2 \left(\int_{R^3} \beta_{jk}(x_j - x_k)V(x_j - x_k)\omega_r^2(x_N)dx_N \right) d\tilde{x}$$
$$= o(1).$$

Consequently, in $r \to \infty$ we have

$$(H_N\psi, \psi) = (H_{N-1}\varphi, \varphi) + o(1)$$

Thus
$$E_N = \inf_{||\psi||=1} (H_N\psi, \psi) \leqslant (H_{N-1}\varphi, \varphi) + o(1)$$

Hence
$$E_N \leqslant \inf_{||\varphi||=1} (H_N\varphi, \varphi) = E_{N-1}$$

Theorem 1 is proved.

Proof of Theorem 2.

Note that if the inequality $E_N \leq E_{N-1}$ holds for some number N, then the Theorem 2 follows directly from Theorem 1. To prove this inequality for some number N we divide the space R^{3N} as follows. Fix a number $\rho > 0$ Ð¸ δ, $0 < \delta < \frac{1}{2}$,and let $d(x) = \max_{1 \leqslant j \leqslant N} |x_j|$.

Let us introduce the following sets

$$A_0 = \{x \in R^{3N} : \ |x_j| < \rho, \ \text{j} = 1,2,...,\text{N}\},$$

$$A_i = \{x \in R^{3N}: \ |x_i| > (1-\delta)d(x), \ d(x) > \frac{1}{2}\delta\},$$
$$i = 1, 2, ..., N.$$

Let $\{J_i\}_{i=1}^{N}$ - a partition of unity with supp $J_i \subset A_i$ such that

$$\sum_{i=0}^{N} \ |\nabla J_i(x)| \ ^2 \ \leqslant \ \frac{AN^{\frac{1}{2}}}{\rho^2} \ , x \in \ A_0,$$

$$\sum_{i=0}^{N} \ |\nabla J_i(x)| \ ^2 \ \leqslant \ \frac{AN^{\frac{1}{2}}}{d(x)\rho} \ , x \in \ A_j, \ j \geqslant 1,$$

where A - a constant. Existence of such a partition proved in (Cycon et al., 1987). With such a partition an operator H_N can be represented in the following way:

$$H_N = J_0(H_N - L(x))J_0 + \sum_{i=1}^{N} J_i(H_N - L(x))J_i,$$

where

$$L(x) = \sum_{i=0}^{N} \ |\nabla J_i(x)| \ ^2$$

called the localization error. This representation is known as IMS-localization formula (Cycon et al., 1987). Let us now estimate the first term.

$$J_0(H_N - L)J_0 \geqslant$$

$$\geqslant \ J_0 \left(\sum_{i=1}^{N} (-\Delta_i + \alpha_j(x_j)V(\ x_j \ )) \right) J_0 +$$
$$+ J_0 \left(\sum_{k=1}^{N} \sum_{j=1}^{k-1} \beta_{jk}(x_j - x_k)V(x_j - x_k) - A\frac{N^{\frac{1}{2}}}{\rho^2} \right) J_0.$$

Since $\alpha_j(x_j) \leqslant Z$ and for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ for an arbitrary function $\varphi \in C_0^{\infty}(A_0)$:

$$(V(x_j)\varphi, \ \varphi \,) \leqslant \ \ \varepsilon(-\Delta\varphi, \varphi \,) + \ \delta(\varepsilon)(\varphi, \varphi)$$

then there exist a constant $c > 0$ such that

$$J_0 \left(-\sum_{j=1}^{N} (\Delta_j + \alpha(x_j)V(x_j)) \right) J_0 =$$

$$= J_0 \left(-\sum_{j=1}^{N} \Delta_j - \sum_{j=1}^{N} \alpha(x_j)V(x_j) \right) J_0 \geqslant cNZJ_0^2.$$

Moreover, $V(x_j - x_k) \geqslant V(2\rho), \;\; x_j \in A_0$ hence

$$\sum_{k=1}^{N} \sum_{j=1}^{k-1} \beta_{jk}(x_j - x_k)V(x_j - x_k) \geqslant$$

$$\geqslant V(2\rho) \sum_{k=1}^{N} \sum_{j=1}^{k-1} \beta_{jk}(x_j - x_k) =$$

$$= V(2\rho)(N-1) \sum_{k=1}^{N} \sum_{j=1}^{k-1} \sigma_2(k, \beta) > \frac{V(2\rho)}{\nu} N(N-1).$$

Then, for large N

$$J_0(H_N - L(x))J_0 \geqslant$$

$$\geqslant J_0 \left(-cZN + \frac{V(2\rho)}{\nu} N(N-1) - \frac{A}{\rho^2} N^{\frac{1}{2}} \right) J_0 \geqslant 0.$$

$$J_0(H_N - L)J_0 \geqslant$$

$$\geqslant \;\; J_0 \left((-Nc(Z) + \frac{N(N-1)}{2} V_2(2\rho) - \frac{AN^{\frac{1}{2}}}{\rho^{\;2}} \right) J_0 \geqslant 0.$$

Denoted by $H_{N-1}^{(i)}$, where $i \neq 0$, an operator

$$H_{N-1}^{(i)} = -\sum_{\substack{j=1 \\ j \neq i}}^{N} (\Delta_j + \alpha_j(x_j)V(x_j)) +$$

$$+\sum_{\substack{k=1\\k,j\neq i}}^{N}\sum_{j=1}^{k-1}\beta_{jk}(x_j-x_k)V(x_j-x_k)$$

For any $x_i \in A_i$

$$J_i(H_N-L(x))J_i \geq$$
$$\geq J_i\left(H_{N-1}^{(i)}-\Delta_i-\alpha_i(x_i)V(x_i)\right)J_i+$$
$$+J_i\left(\sum_{\substack{j=1\\j\neq i}}^{N}\beta_{ji}(x_j-x_i)V(x_j-x_i)-\frac{A}{d(x)\rho}N^{\frac{1}{2}}\right)J_i$$

Clearly

$$V(x_i-x_j)\geqslant\frac{b(2d(x))}{2d(x)},$$
$$V(x_i)\geqslant\frac{b(d(x))}{d(x)}.$$

Then

$$J_i(H_N-L(x))J_i \geq$$
$$\geq J_i\left(E_{N-1}+\frac{b(|x_i|)}{|x_i|}(-\alpha_i(x_i)\right)J_i+$$
$$J_i\left(\frac{b(2d(x))}{2d(x)}\frac{|x_i|}{b(|x_i|)}\sum_{\substack{j=1\\j\neq i}}^{N}\beta_{ji}(x_j-x_i)-\frac{A}{d(x)\rho}N^{\frac{1}{2}}\frac{|x_i|}{b(|x_i|)})\right)J_i$$

Since $b(x)$ a non-decreasing function, for $x \in A_i$

$$\frac{b(2d(x))}{2d(x)}\cdot\frac{|x_i|}{b(|x_i|)}=\frac{b(2d(x))}{b(|x_i|)}\cdot\frac{|x_i|}{2d(x)}\geqslant\frac{1-\delta}{2};$$

and

$$\sum_{j=1,j\neq i}\beta_{ji}(x_j-x_i)=2(N-1)\sigma_2(i,\beta)\geqslant\frac{2}{\nu}(N-1);$$

$$\frac{|x_i|}{d(x)b(|x_i|)} \leqslant \frac{1}{b\left(\frac{1-\delta}{2}\rho\right)}.$$

then

$$J_i(H_N - L(x))J_i \geqslant$$

$$\geqslant J_i\left(E_{N-1} + \frac{b_1(|x_i|)}{|x_i|}(-Z + (N-1)\frac{1-\delta}{\nu} - \frac{AN^{\frac{1}{2}}}{b\left(\frac{1-\delta}{2}\rho\right)\rho}\right)J_i.$$

Thus, for large N, the inequality

$$E_N \leqslant E_{N-1}$$

holds. Theorem 2 is proved.

Proof of Theorem 3.

Assume that $E_N < E_{N-1}$ and let H^k_{N-1} the Hamiltonian of the system without a particle x_k, i.e., for any fixed $k\,(1 \leqslant k \leqslant N)$

$$H^{(i)}_{N-1} - \Delta_i - \alpha(x_j)V(x_j) + \sum_{j\neq k}\beta_{jk}(x_j - x_k)V(x_j - x_k)$$

Now let $H_N f = E_N f$ and $(f, f) = 1$. Then

$$0 = (V^{-1}(x_k)f, (H_N - E_N)f) =$$

$$= (V^{-1}(x_k)f, (H^k_{N-1} - E_N)f) - (V^{-1}(x_k)f,\ \Delta_k f) - (\alpha_k(x_k)f, f) +$$

$$+(V^{-1}(x_k)f\ ,\ \sum_{j\neq k}\beta_{jk}(x_j - x_k)\,V(x_j - x_k)f).$$

Now taking into account the inequalities $H_{N-1} \geq E_{N-1} > E_N$ and applying Lemma 2 of (Cycon et al., 1987), we obtain

$$(V^{-1}(x_k)f\,,\ \sum_{j\neq k}\beta_{jk}(x_j-x_k)\,V(x_j-x_k)f) < (\alpha_k(x_k)f,f)\,.$$

We sum these inequalities for $k = 1, 2, ..., N$:

$$\sum_{k=1}^{N}\left(V^{-1}(x_k)f\,,\ \sum_{j\neq k}\beta_{jk}(x_j-x_k)\,V(x_j-x_k)f\right) < \left(\sum_{k=1}^{N}\alpha_k(x_k)f,f\right),$$

or equivalently

$$\sum_{j=1}^{N}(V^{-1}(x_j)f,\sum_{k\neq j}\beta_{kj}(x_k-x_j)V(x_k-x_j)f) < (\sum_{k=1}^{N}\alpha_k(x_k)f,f).$$

Summing the last two inequalities and considering that

$$\beta_{kj}(x_k-x_j)V(x_k-x_j) = \beta_{jk}(x_j-x_k)V(x_j-x_k)$$

obtain

$$\sum_{k=1}^{N}\sum_{j\neq k}\left(\left(V^{-1}(x_k)+V^{-1}(x_j)\right)V(x_j-x_k)\beta_{jk}(x_j-x_k)f,f\right) < 2\left(\sum_{k=1}^{N}\alpha_k(x_k)f,f\right)$$

Hence, by the corollary to Lemma 3 in [1], we obtain

$$\sum_{k=1}^{N}\sum_{j\neq k}(\beta_{jk}(x_j-x_k)f,f) < 2\left(\sum_{k=1}^{N}\alpha_k(x_k)f,f\right)$$

or

$$(\sum_{k=1}^{N}\sum_{j<k}\beta_{jk}(x_j - x_k)f, f) < (\sum_{k=1}^{N}\alpha_k(x_k)f, f)$$

and because of (1) we have

$$N - 1 < 2\nu Z$$

Theorem 3 is proved.

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